the correction procedure has not been verified in that range.

Figure 3 shows good agreement between the measured and statistically calculated heat capacity of tetrafluoromethane over the 30° to 200°C. range. In view of this agreement the statistical calculations were extended over a much wider range, 140° to 620°K., and the data fitted to an equation

$$C_V = 0.00375 + \frac{102.9}{T^2} + 5.7857 \times 10^{-4}T - 3.7829 \times 10^{-7}T^2 \text{ cal./g.} - {}^\circ\text{K.} \quad (4)$$

with T in °K. This fits the data with a maximum error of only 0.3%. It is worthwhile mentioning that a curve of C_V^* vs. T undergoes an inflection which cannot be represented precisely with a cubic expansion in positive powers of T as is usually done. It is necessary to use an expansion in reciprocal temperature or some other more complex mathematical function.

ACKNOWLEDGMENT

The authors are grateful to the E. I. du Pont de Nemours and Company for financial assistance and for supplying the samples of tetrafluoromethane, to the Esso Research and Engineering Company for an equipment grant, and to The University of Michigan Rackham Graduate School for fellowship support.

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Manuscript received March 15, 1963; revision received June 18, 1963; paper accepted June 27, 1963. Paper presented at A.I.Ch.E. New Orleans meeting.

Idealized Theory for Turbulent Mixing in Vessels

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The engineer commonly estimates the influence of mixing on process operations by resorting to useful simplifying assumptions. Two such assumptions are the notions of plug flow and well-mixed systems. In the following treatment a means is demonstrated for predicting states of mixing for miscible systems which are intermediate to the above mentioned cases. This analysis utilizes mostly known results of the modern theory of turbulence as based on the statistical description and exploits Kolmogoroff's concept of universal equilibrium (*10*). The system studied is the continuous flow stirred vessel. Current state of the art regarding such turbulent mixers allows one to predict the power input given knowledge of the mixer geometry, fluid properties, and stirring rate. Power input rates are well correlated for many existing mixers through work such as that of Rushton et al. (*12*). What has not been adequately known is how to predict the effect of this specified stirring on the mixing job to be done.

The stirred tank mixer characteristically operates up to high Reynolds' numbers based on impeller diameter and tip velocity ($\sim 10^6$) which should be conducive to approaching universal equilibrium in the Kolmogoroff sense (*10*). Shinnar (*13*) has advanced some stimulating conjectures on the nature of emulsion formation in stirred tanks in which he exploits behavior of the velocity spectrum at wave numbers in the dissipative range of eddy sizes. In the present work knowledge of the inertial subrange, a lower wave number region, is exploited.

The dissipation of scalar inhomogeneities is generally pictured to be preceded by cascade down the spectrum.

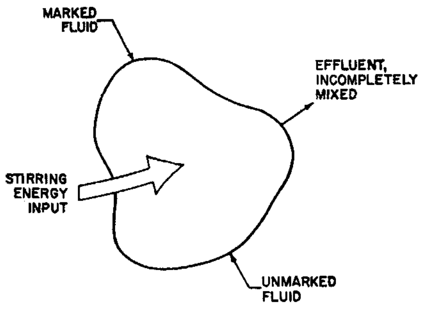

Fig. 1. Schematic of generalized mixer. Stirring energy input is by means of mechanical or fluid stirring.

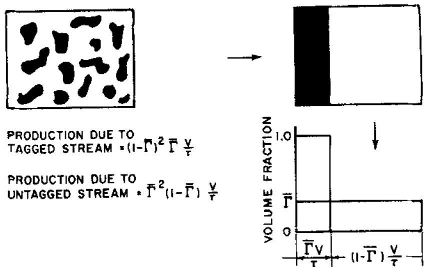

Fig. 2. Scheme for deriving conservation of $\int_v \gamma^2 dV$.

Input of mean square scalar fluctuation content to the low end of the spectrum then equals throughput in a universal subrange where molecular diffusion is not appreciable, this followed by transfer to still smaller eddy sizes where dissipation by molecular diffusion predominates. Thus knowledge of the cascade rate is equivalent to knowledge of the dissipation rate. It is found expedient in this work to base the analysis on cascading. This may be compared with Corrsin's (*4*) attack on the batch mixer which is crucially based on behavior at high wave numbers and for which knowledge of spectrum appears to be not as complete, although more recent contributions by Batchelor, Howells, and Townsend (*8* page 234) have clarified this situation to an extent.

CONSERVATION STATEMENT

The system, one of continuous flow, is schematized in Figure 1. A marked stream enters at the volumetric rate $\overline{\Gamma}V/\tau$ and an unmarked stream at the volumetric rate $(1-\overline{\Gamma})V/\tau$. They are stirred in a turbulent fashion, and the partially mixed contents of the tank are drawn off at the volumetric rate V/τ. A simplifying assumption is that all portions of the tank contents have, in the statistical sense, the same unmixedness. This condition requires a large eddy diffusivity, although the concept of an eddy diffusivity plays no further role in the treatment. The descriptive term "unmixedness" was first coined by Hottel (*7*) and co-workers in early work on flame turbulence. The unmixedness is herein identified as γ', the root-mean-square fluctuating concentration sometimes referred to as the *intensity*. By the assumption of this model the average over the holdup volume at one instant will be the same as the time average at a given point.

There are essentially three mechanisms at work to produce the unmixedness found in the tank. There is production of unmixedness due to introduction of feed streams, output of unmixedness as carried by the effluent stream, and dissipation of unmixedness by fine scale mixing. These will be treated in turn. The procedure is to obtain a conservation statement for the entity $\int_V \gamma^2\, dV$.

PRODUCTION OF MEAN-SQUARE UNMIXEDNESS

To picture the manner in which the input stream creates unmixedness suppose there is no molecular diffusion mixing in the tank. As sketched in Figure 2 blobs of each species will be interdispersed, and each blob will be bounded with sharply defined edges. The local composition of a species will be either zero or unity in volume fraction at any given point, and for such a distribution of matter one can at once calculate $\overline{\gamma'^2}$. In one unit of time a volume numerically equal to $\overline{\Gamma}V/\tau$ of the tagged stream enters along with volume $(1-\overline{\Gamma})V/\tau$ of the untagged stream. The tagged stream has a volume fraction of unity and so contributes $(1-\overline{\Gamma})^2\,\overline{\Gamma}V/\tau$ towards a total squared deviation from the mean concentration, while the other stream, with zero concentration, contributes $\overline{\Gamma}^2\,(1-\overline{\Gamma})\,V/\tau$. Hence the total production in unit time is the sum of these terms or $\overline{\Gamma}\,(1-\overline{\Gamma})\,V/\tau$. Reflection will indicate that this production rate will be the same whether molecular mixing occurs or not.

OUTPUT OF UNMIXEDNESS

The contents of this highly idealized tank are assumed to be homogeneously inhomogeneous so that they and the effluent stream have the same unmixedness. Denoting the unmixedness for a given state of operation as γ'^2 one may write at once, for the output rate, $\gamma'^2\, V/\tau$.

DISSIPATION OF UNMIXEDNESS

The turbulent velocity field grinds down the continually introduced concentration blobs into a spectrum of finer sizes. Beyond some small size a blob cannot be detected by a given measuring technique and may be considered mixed. This size corresponds to a characteristic lineal dimension of the observed point. This situation arises, for instance, in the smoke-scattered light mixing experiments reported earlier (*11*). For the smoke-scattered light technique molecular diffusion distances for aerosol particles corresponding to flow system residence times are generally small compared with optical probe dimensions, and so no appreciable Brownian diffusion takes place. Nonetheless the experimentally determined unmixedness decreases. It is clear that no discussion based on the molecular or Brownian diffusion coefficients is applicable to describe the measured phenomenon and so the notion of spectral cascade will be drawn on. Later it will become apparent that this treatment has utility in describing mixing of true molecular species as well. The spectral cascade process for concentration was introduced by Corrsin (*2*) and is discussed by Hinze (*8* page 232). It is assumed that large concentration eddies pass through a succession of progressively smaller sizes and with no dissipation. In universal equilibrium there is by definition a range of eddy sizes unaffected in their unmixedness content by the flow apparatus. This will not be assumed for a moment though, and the rate of transfer down the spectrum, in or out of universal equilibrium, is denoted by ϵ_γ on the unit volume basis. This transferred unmixedness is numerically the dissipation rate by molecular mixing or otherwise there would not be steady state as assumed. The tank containing a volume V of holdup will thus dissipate at the total rate of $\epsilon_\gamma V$.

The rate of production is now set equal to the rate of output plus the rate of dissipation of scalar inhomogeneities, and the result after minor rearrangement may be expressed as follows:

$$\overline{\gamma'^2} = (1-\bar{\Gamma})\,\bar{\Gamma} - \epsilon_\gamma \tau \qquad (1)$$

This equation relates a measure of γ'^2, unmixedness, to composition, residence time, and specific dissipation rate. It predicts correct results for the limiting cases of $\epsilon_\gamma \to 0$, $\tau \to 0$, and single species flow, that is $\bar{\Gamma} \to 0$ or 1. The equation would provide a basis for computing concentration energy cascade rate from experimental values of concentration fluctuation, residence time, and average concentration. The next section however attacks the problem of predicting ϵ_γ from fundamental considerations.

DETERMINATION OF ϵ_γ

Refined reasoning has in the past lead to the guess that $G_1(k_1)$, the unnormalized spectral distribution of concentration, is a function only of ϵ_γ, $g_c\epsilon$, and k_1 for a certain range of k_1 values. The k_1 values so included are referred to as the *inertial-convective subrange*. Corrsin (2) obtained the following result on the basis of dimensional analysis:

$$G_1(k_1) = C_\gamma\, \epsilon_\gamma\, (g_c\epsilon)^{-1/3}\, k^{-5/3} \qquad (2)$$

Equation (2) contains a constant C_γ which supposedly has a universal numerical value. The evidence for that will be summarized below. The interesting point is that the equation relates ϵ_γ to other system parameters:

$$\epsilon_\gamma = \frac{G_1(k_1)\,(g_c\epsilon)^{1/3} k_1^{5/3}}{C_\gamma} \quad \text{(inertial subrange only)} \qquad (3)$$

Actual spectra turn out to be described by Von Karman's interpolation formula. A form due to Hinze (8) is expressed in terms of unmixedness and a characteristic wave number k_o representative of the large concentration eddies:

$$G_1(k_1) = \overline{\gamma'^2}\, \frac{2\Gamma(5/6)}{\sqrt{\pi}\,\Gamma(1/3)\,k_o} \left[1 + \left(\frac{k_1}{k_o}\right)^2\right]^{-5/6} \qquad (4)$$

The numerical constant is expressed in terms of the gamma function. This expression has behavior for $k_1 \to 0$ appropriate to true isotropic turbulence, though this can probably never be achieved in practice. The experimental data from jets, wakes, enclosed mixers, and wind tunnels are fairly well correlated by Equation (4) over the range of fluctuation-containing eddies. For larger wave numbers Equation (4) reduces to a —5/3 law statement appropriate to an inertial subrange. It fails on theoretical grounds since it shows dependence on molecular properties of viscosity or diffusivity at wave numbers beyond the inertial subrange, but this is unimportant for the present problem. Again the reason for this is that the molecular dissipation range of the spectrum has an existence which is influenced by the processes in the cascade portion of the spectrum while the reverse is not so. Hence, since the present treatment is based on a description of the cascade process only, it is unimportant that Equation (4) fails in the higher wave number range.

From $(k_1/k_o) >> 1$ Equation (4) reduces to

$$G_1(k_1) = \frac{2\Gamma(5/6)}{\sqrt{\pi}\,\Gamma(1/3)}\, \overline{\gamma'^2}\, k_o^{2/3}\, k_1^{-5/3} \qquad (5)$$

Comparison of Equations (2) and (5) yields a relationship for dissipation rate expressed as a function of unmixedness, large eddy size, and mechanical energy dissipation rate:

$$\epsilon_\gamma = \frac{2\Gamma(5/6)}{\sqrt{\pi}\,\Gamma(1/3)}\, \frac{\overline{\gamma'^2}\, k_o^{2/3}\,(g_c\epsilon)^{1/3}}{C_\gamma} \qquad (6)$$

Equation (6) states that ϵ_γ increases with energy dissipation rate, a reasonable trend, and with $\overline{\gamma^2}$, also a reasonable trend.

As a relatively simple alternative analysis it may be assumed that the dissipation is proportional to the fluctuations $(\overline{\gamma'^2})$ and dependent on a characteristic length $(1/k_o)$ and the rate of energy dissipation $(g_c\epsilon)$. Dimensional analysis then leads directly to Equation (6), although the numerical constant of proportionality is then not determined.

The wave number k_o is related to integral scale. For integral scale Λ_γ defined as *area* under the correlation coefficient curve [see (8) p. 235]

$$\Lambda_\gamma = \frac{\sqrt{\pi}\,\Gamma(5/6)}{\Gamma(1/3)\,k_o} = \frac{0.75}{k_o}$$

Also the peak in the three-dimensional spectrum corresponding to Equation (4) is at $k_p = \sqrt{(6/5)}\ k_o$. Hence

$$\Lambda_\gamma = 0.75\sqrt{6/5}\,\frac{1}{k_p} = \frac{0.82}{k_p}$$

and one may nearly synonymously speak of the scales k_o^{-1}, k_p^{-1}, and Λ_γ.

In general it would appear that any one of these scale parameters would be a function of the dimensions of the vessel, the stirrer, and the inlet port. For orientation purposes and in all that follows it will be assumed sufficient to assume proportionality between k_o and $1/d$, where d is the hydraulic radius of an inlet. Choosing the scale factor to correspond to inlet port dimensions is a situation difficult to imagine in certain cases, for example in a well-baffled, stirred tank. Hence this choice of scale factor is really an order of magnitude consideration. Equation (6) then indicates that smaller feed ports favor higher mixing rates, since k_o is in the numerator to a positive power.

PREDICTION OF UNMIXEDNESS

An explicit expression for unmixedness now follows by combining Equations (6) and (1):

$$\frac{\overline{\gamma'^2}}{(1-\bar{\Gamma})\bar{\Gamma}} = [1 + N(k_o^2\, g_c\epsilon)^{1/3}\tau]^{-1} \qquad (7)$$

where

$$N = \frac{2\Gamma(5/6)}{\sqrt{\pi}\,\Gamma(1/3)\,C_\gamma} \qquad (8)$$

and where $\Gamma(x)$ is the gamma function as before. Equation (7) provides a rational means for scaling or designing mixers to perform a specified amount of mixing. It indicates the influence of feed rate, holdup, and energy input.

DETERMINATION OF THE CONSTANT C_γ

Wind Tunnel

The data for this calculation may be gotten from the wind tunnel work of Kistler, O'Brien, and Corrsin (9). In their experiments a grid of heating wires was introduced a small distance upstream of a dowel grid. In that way the hot air passed through the turbulence-generating shear zones of the momentum grid so that both temperature and velocity fluctuations were produced in the same local neighborhoods. The random fields decay as they are carried downstream, and this process is uncomplicated by any appreciable production effects such as would be present with mean gradients acted upon by turbulent diffusion. As a result the decay rate may be derived from the slope of the intensity-distance plot shown in Figure 3. Those authors calculated microscales with such slope

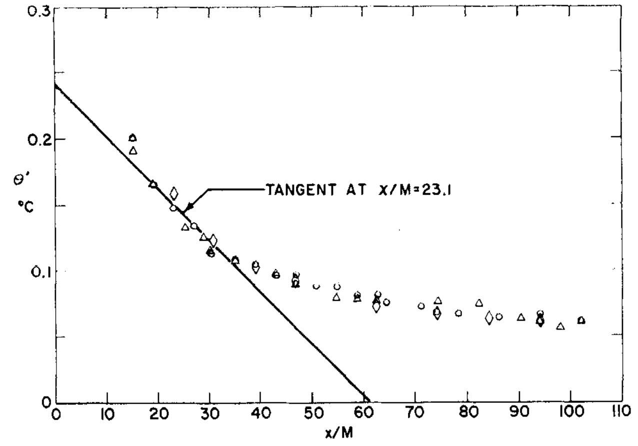

Fig. 3. Temperature fluctuation decay in wind-tunnel turbulence (Data of Kistler, O'Brien, and Corrsin, reference 9).

data using the following equations in which θ is root-mean-square temperature fluctuation:

$$\epsilon_\theta = \frac{d\theta^2}{dt} = -12\,\alpha \frac{\theta^2}{\lambda^2_\theta} \quad (9)$$

$$\frac{2}{3}(g_c \epsilon) = \frac{du^2}{dt} = -10\nu \frac{u^2}{\lambda^2} \quad (10)$$

This gave numerical values of $\lambda = 0.013$ ft. and $\lambda_\theta = 0.015$ ft. for the station at $x/M = 23$. With kinematic viscosity $\nu = 1.6 \times 10^{-4}$ sq. ft./sec. and Prandtl number (ν/α) of 0.7 one can back calculate values of $\epsilon_\theta = 0.28$ $(°C)^2$/sec. and $\frac{2}{3}\,\epsilon\, g_c = 1.0$. Figure 4 which is redrawn from Figure 8 of reference 9 is a plot of log spectrum, that is log $G_1(k_1)$, against log wave number k_1 from which it is determined that there is a little less than one decade of wave numbers that defines a slope of $-5/3$ and that range is centered about k_1 of 70 ft.$^{-1}$. This is not extensive but there seems to be no definite reason why it need be in order to be considered as an inertial-convective subrange. It is encouraging that a plot of $k_1^2\, G_1(k_1)$, a measure of the dissipation spectrum, proves to have a considerable portion of its content beyond the inertial-convective subrange identified above. For any point within this inertial-convective subrange one has $k_1^{5/3}\, G_1(k_1) = 0.106$ $(°C)^2$ ft.$^{-2/3}$. Then in accordance with Equation (3) the numerical value of the constant is

$$\begin{aligned} C_\gamma &= k_1^{5/3}\, G_1(k_1)\, (g_c\epsilon)^{1/3}\, \epsilon_\theta^{-1} \\ &= (0.106)\,(3/2)^{1/3}\,(0.28)^{-1} \\ &= 0.43 \end{aligned}$$

For a three-dimensional spectrum the corresponding constant is 5/3 as large or 0.72 provided Equation (4) has the correct dependence on wave number. An estimate of order unity for C_γ has been obtained on theoretical grounds (2, 3) with a method of von Neumann's that appeals to an asymptotic form of the spectrum at very large Reynold's numbers; hence the empirically determined number is quite reasonable.

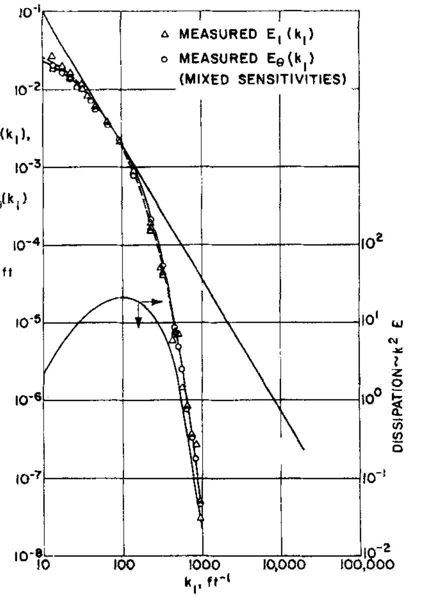

Fig. 4. Velocity and temperature fluctuation spectra in wind-tunnel turbulence at x/M = 23.1 (Data of reference 9).

Warm Jet

Corrsin and Uberoi (5) have measured spectrum and microscale in a heated jet, and from their data it is possible to obtain another experimental value of the hopefully universal constant C_γ. The relevant parameters are in Table 1, and the calculation yields $C_\gamma \simeq 0.2$. This agreement of the two calculated values of C_γ is encouraging.

Smoke Jet

Rosensweig, Hottel, and Williams (*11*) report concentration field measurements determined with an optical probe in a free jet. The normalized concentration fluctuation spectrum is identical with the warm jet to good precision, but the measured intensity is higher everywhere and about 50% higher at the center line as shown in Figure 5. This is believed due to more accurate response to large fluctuations for the optical probe technique. From Equation (3) it is seen that, all other factors being equal, C_γ is directly proportional to the mean-square fluctuation level so that $C_\gamma \simeq 1.5$ (0.18 or 0.23) = 0.26 or 0.33.

ILLUSTRATIVE EXAMPLES

Mechanically-Stirred Mixer

A tank with a holdup of 50 gal. is fed with two incompressible fluid streams of density 62.3 lb.$_m$/cu. ft. which are blended by a flat-bladed turbine driven with a motor that supplies 1 hp. to the impeller. The streams enter at equal rates through two 3 in. I.D. pipes at one side of

TABLE 1. EVALUATION OF C_γ

	x/d or x/M	u', ft.-sec.$^{-1}$	$\overline{U}$, ft.-sec.$^{-1}$	θ, °C.	λ, ft.	λ_θ, ft.	$\frac{du'^2}{dt}$, sq. ft.-sec.$^{-3}$	ϵ_γ, $\frac{d\gamma'^2}{dt}$, (°C.)^2-sec.$^{-1}$	C_γ
Wind tunnel	23	0.31	14	0.15	0.0125	0.015	0.99	0.28	0.38
Jet	20	8.9-12.6	135	5.5	0.0092	0.014	1,530	423	0.18-0.23
Theory									0(1)
Jet (corrected)									0.26-0.33

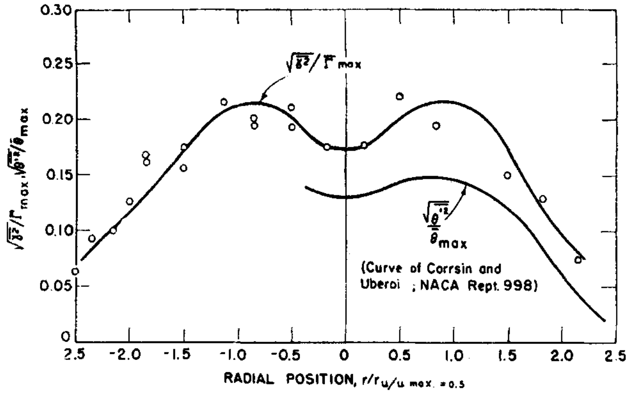

Fig. 5. Scalar fluctuations in turbulent, free, round jet at 15 nozzle diameters downstream. Data points are from reference *11*.

the tank, and effluent mixture is drawn off at the opposite side. One stream is dyed red, and the other is clear. Compute the unmixedness of the effluent stream for entering velocities of 5 ft./sec.

The problem of determining what fraction of the input mechanical energy is converted to turbulent fluctuations in the velocity field which subsequently cascade down the spectrum is not intended to be treated in this paper. However it is bracketed between zero and unity, and for the present purposes it will be assumed to be 50%. The dimension of eddies related to the peak in the three-dimensional spectrum may be assumed the same as the diameter of an inlet pipe. The universal constant C_γ is estimated to be 1/3 on the basis of the experimental data presented above.

Solution. Compute numerical values of use in Equation (7).

One horsepower is 550 ft.-lb.$_f$/sec. The tank holds 50 (8.33) = 416. lb.$_m$. Hence

$$\epsilon = \frac{0.5\ (550)}{416.} = 0.66\ \frac{\text{ft.-lb.}_f}{\text{sec.-lb.}_m}$$

$$g_c = 32.174\ \frac{\text{lb.}_m\ \text{ft.}}{\text{lb.}_f\ \text{sec. sq.}}$$

$$k_o = \frac{12}{3} = 4 \text{ cycles/ft.}$$

$$V = \frac{50}{7.48} = 6.69 \text{ cu. ft.}$$

$$N = \frac{2\Gamma(5/6)}{\sqrt{\pi}\,\Gamma(1/3)\ C_\gamma} = \frac{0.48}{C_\gamma} = 1.44$$

Inlet flow rate $= \frac{\pi}{4}\left(\frac{3}{12}\right)^2$ (5) $=$ 0.25 cu. ft./sec. from one stream

$$\tau = \frac{6.69}{(0.25)\,(2.0)} = 13.4 \text{ sec.}$$

Thus, from Equation (7) and for $\overline{\Gamma} = 0.5$

$$\frac{\gamma'^2}{\overline{\Gamma}} = \frac{\gamma'^2}{(1-\overline{\Gamma})\overline{\Gamma}} = [1 + 1.44\ (4^2 \times 32.174 \times 0.66)^{1/3} \times 13.4]^{-1}$$

from which $\gamma'/\overline{\Gamma} = 0.086$. The product stream is predicted to have an unmixedness of 8.6%.

In general the inlet streams will also contribute to the stirring, and the corresponding additional energy input should be added in. In this example the kinetic energy of entering streams is $\frac{(5)^2(0.50)\,(62.3)}{2(32.174)\,(550)} = 0.023$ hp. and hence is negligible. In the next example an opposite limiting case is examined.

Fluid-Stirred Mixer

A spherical chamber of diameter D is fed two streams, each at the same volumetric rate $\overline{\Gamma}V/\tau$ which enter from opposed feed ports of diameter d in the wall. There is general good recirculation within the mixer, and the mixture is continually exhausted with negligible velocity from a large number of exit ports uniformly spaced about the wall. Analyze this situation.

Solution. The kinetic energy of the entering streams is dissipated to heat. Some of this dissipation is direct owing to existence of gradients in the mean velocity field while the remainder, say a fraction η, may be assumed to cascade. Then $g_c\epsilon = \left(\frac{u^2\,\eta}{V}\right)\left(\frac{\overline{\Gamma}V}{\tau}\right)$, where $V = \pi D^3/6$ is the spherical volume and u is the entering velocity. Thus

$$g_c\epsilon = \left(\frac{16\eta}{\pi^2\,d^4\,V}\right)\left(\frac{\overline{\Gamma}V}{\tau}\right)^3 \qquad (11)$$

Substituting into Equation (7), assuming $k_o = 1/d$ and $N = 1.44$ as before, one obtains with $\overline{\Gamma} = 0.5$

$$\frac{\gamma'}{\overline{\Gamma}} = \left(1 + 0.55\,\eta^{1/3}\,\frac{D^2}{d^2}\right)^{-1/2} \qquad (12)$$

This result says the extent of mixing in this case of equal flow rates depends only on geometrical factors and not at all on fluid properties or flow rate, excepting small variations in $\eta^{1/3}$. For $\eta = 0.5$ and diameters in the proportion of 10 to 1 the prediction is $\gamma'/\overline{\Gamma} \simeq 15\%$. Of course the mean concentration both in the reactor and at the outlet is 0.5, expressed as a volume fraction of one of the feed-stream compositions. It should be noted that use of Equation (7) is not restricted to equal flow rates. Furthermore the scale factor related to k_o for more general inlet port geometries might possibly be related to the hydraulic diameter.

Comparison of Theory with Experimental Data

Becker (*1*) has measured the concentration fluctuations in the ducted jet sketched in Figure 6 consisting of an 8-in. diameter tube 50 in. in length into which is fed at one end and on the center line the discharge from a ¼-in. round nozzle at 430 ft./sec. Inspired air enters through the duct cross-sectional area surrounding the nozzle and is controlled by throttling screens. The flow is characterized by a throttling parameter N_{Th}, and recirculating flow within the duct is observed for $N_{Th} \leq 0.43$. The leading edge of the recirculation eddy is found to be 10 N_{Th} duct radii downstream. The throttling number is defined in terms of nozzle velocity u_N, average velocity of inspired fluid u_I which enters over a cross section of radius r_D, nozzle radius r_N, and duct radius r_D:

$$N^2_{Th} = \frac{\left[\frac{u_N}{u_I}\left(\frac{r_N}{r_D}\right)^2 + 1\right]^2}{2\left(\frac{u_N}{u_I}\right)^2\left(\frac{r_N}{r_D}\right)^2 + 1} \qquad (13)$$

The average composition of fluid corresponding to the input flows is

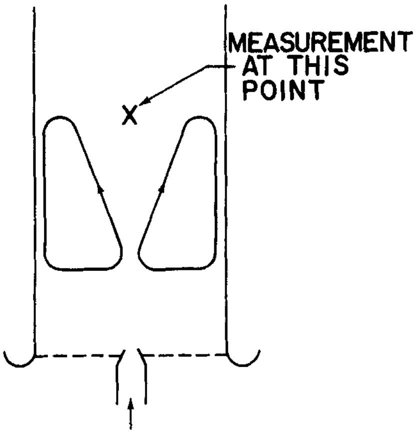

Fig. 6. Ducted jet mixer of Becker (1) sketched to scale. Measurement is performed downstream of the recirculation region.

$$\bar{\Gamma} \simeq \frac{u_N\, r_N^2}{u_I\, r_D^2 + u_N\, r_N^2} = \frac{\dfrac{u_N}{u_I}\left(\dfrac{r_N}{r_D}\right)^2}{1 + \dfrac{u_N}{u_I}\left(\dfrac{r_N}{r_D}\right)^2} \qquad (14)$$

Solving (14) for the velocity ratio and substituting the result into (13) one obtains

$$\frac{1}{N^2_{Th}} = \left(1 + 2\frac{r_D^2}{r_N^2}\right)\bar{\Gamma}^2 - 2\bar{\Gamma} + 1 \qquad (15)$$

The ratio r_D/r_N was 30.8 in the experiments, and from the above equation the values of $\bar{\Gamma}$ in Table 2 were calculated corresponding to the given values of *Th*. From the volumetric flow rate through the nozzle (about 0.15 cu. ft./sec.) and the values of $\bar{\Gamma}$ it is possible to calculate the total volumetric flow rates $\dot{v}_T$ in column three. Experimental measurements on the center line 13.45 in. downstream from the nozzle are used for this analysis. That location defines a mixer volume V of 0.39 cu. ft. Values of residence time are then calculable as $\tau = V/\dot{v}_T$ and are listed in column four. The experimental intensities $\gamma'/\bar{\Gamma}$ in column five are from Figure 8.3.1 of Becker. The last column gives calculations of the left-hand side of Equation (7). In accordance with Equation (7) the variable $\dfrac{\bar{\Gamma}(1-\bar{\Gamma})}{\gamma'^2}$ plotted vs. τ should lie on a straight line intersecting the ordinate at unity and having a slope of $N(k_o^2\, g_c\epsilon)^{1/3}$. It is seen from Figure 7 that the points are ordered with respect to the throttling parameter N_{Th} and that a straight line provides a good fit to the three data points and unity on the ordinate, thus showing that these data are consistent with the theory. The numerical value of the slope provides an estimate of the scale if the energy dissipation rate is known. An estimate of the energy dissipation rate will next be obtained. The rate kinetic energy fed into the system is calculable from nozzle-fluid velocity, density, and mass flow rate. The rate of kinetic energy addition in the secondary stream and removal in the outflow is negligible in comparison. For air density of 0.078 lb.m/cu. ft. a ¼-in. diameter nozzle and discharge velocity of 430 ft./sec. there corresponds a total dissipation rate of $\frac{\pi}{4}\left(\frac{1/4}{12}\right)^2 (430)^3\ (0.078)\ (½) = 1050$ ft.-lb.f/sec. With the system volume of 0.39 cu. ft. there corresponds $\frac{g_c\epsilon}{\eta} = \frac{1{,}050}{(0.39)(0.078)} = 34{,}400$ sq. ft./sec. cu. The slope of the line from the graph of Figure 7 is 2,700 sec.$^{-1}$. Thus $N\ (k_o^2\, g_c\epsilon)^{1/3} = 2{,}700$ sec.$^{-1}$ from which $k_o = 610$ cycles/ft. when the fraction of kinetic energy that cascades, η, is set at 0.5. If it is assumed that $k_o = \frac{\beta}{d_N}$, where d is nozzle diameter and β a constant of proportionality, the corresponding numerical value of β is 12.8. Becker's measured values of longitudinal integral scale from spectral measurement (see his Figure 9.5.1) are $0.5 < \frac{\Lambda_\gamma}{d_N} < 1.5$. The corresponding values of β $\left(\beta \simeq 0.75\,\frac{d_N}{\Lambda_\gamma}\right)$ then are bracketed between 1.5 and 0.5 which is in somewhat disappointing agreement with the value of 12.8 deduced above. Mixing conditions in this experimental system can be considered only a very crude approximation to the conditions of the idealized model.

Table 2. Treatment of Experimental Data of Becker

N_{Th}	$\bar{\Gamma}$	$\dot{v}_T$, cu. ft./sec.	τ, sec.	$\gamma'/\bar{\Gamma}$	$\dfrac{\bar{\Gamma}(1-\bar{\Gamma})}{\gamma'^2}$
0.097	0.229	0.640	0.612	0.044	1,740
0.147	0.155	0.945	0.412	0.066	1,260
0.237	0.0942	1.56	0.250	0.154	408

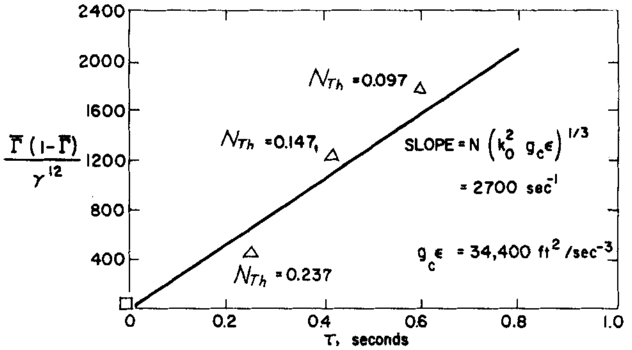

Fig. 7. Experimental mixing data of Becker (1) plotted to test the form of Equation (7).

RANGE OF VALIDITY

This theory is based on a number of simplifying assumptions and a discussion is in order. It has been postulated that the tank contents are homogeneously inhomogeneous; a necessary requirement is existence of good large scale mixing. This in turn would be favored by a high ratio of residence time to recirculation time.

$$\left(\frac{T^3\,\omega\,\tau}{V}\right) >> 1 \qquad \text{(Mechanical Stirring)} \quad (16)$$

$$\left(\frac{T^2\,u\,\tau}{V}\right) >> 1 \qquad \text{(Fluid Stirring)} \quad (17)$$

T represents a characteristic gross dimension of the vessel, ω is impeller angular velocity, and u is an inlet velocity. In addition it is necessary that turbulent mixing be much more intense than molecular mixing, since in the extreme case of infinite molecular diffusivity the contents would always be perfectly mixed even in the absence of turbulence and the theory breaks down. Also existence of an inertial subrange is favored by a large difference in size between the eddies at both ends of the spectrum. The large eddies at the one end are characterized by dimension k_o^{-1}, while the $-5/3$ scalar spectrum will terminate at the other end at the eddy size given roughly by η_v or η_γ, whichever is the larger:

$$\eta_v = \left(\frac{\nu^3}{g_c\epsilon}\right)^{1/4} \quad \eta_\gamma = \left(\frac{D^3}{g_c\epsilon}\right)^{1/4}$$

Which of these is the larger depends in turn on the magnitude of the Schmidt number $N_{Sc} = \nu/D$ since

$$\frac{\eta_v}{\eta_\gamma} = N_{Sc}^{3/4} \tag{18}$$

The Schmidt number is of the order of unity for gases and ranges from 10^2 to 10^4 for liquids. Hence it will generally be sufficient to assume the $-5/3$ portion of the scalar spectrum ends at scale η_v. A necessary condition for existence of the subrange is then

$$(1/k_o) >> \eta \tag{19}$$

Assuming $k_o \sim 1/d$, where d is a dimension of an inlet port, one gets

$$\left(\frac{d^4\, g_c\epsilon}{\nu^3}\right)^{1/4} >> 1 \tag{20}$$

For a fully-baffled and mechanically-stirred system this result may be written in terms of other variables, since it is known that the power group $\frac{\sigma_c\epsilon}{\omega^3 D^2}$ is a function only of the Reynolds number $\left(\frac{D^2\omega\rho}{\mu}\right)$ and, in fact, becomes constant at sufficiently high Reynolds numbers; the D here refers to impeller diameter. At these high Reynolds numbers it follows that $g_c\epsilon$ is proportional to $\omega^3 D^2$, and so the criterion becomes

$$\left(\frac{d^4\, \omega^3\, D^2}{\nu^3}\right)^{1/4} >> 1 \tag{21}$$

or

$$\frac{d}{D} N_{Re}^{3/4} >> 1 \tag{22}$$

This latter condition shows that for the theory to be correct mixing must be intense (high N_{Re}), and also the inlet stream must not be introduced through too small an inlet port d lest molecular mixing dominate before turbulence has a chance to grind down the eddies to smaller sizes.

The recent (6) detection of a $-5/3$ law over more than a three-decade range of wave numbers for extremely high Reynolds numbers turbulence ($N_{Re} \simeq 4 \times 10^7$) offers convincing support to the Kolmogoroff's idea exploited here.

ACKNOWLEDGMENT

The author wishes to thank Professor H. C. Hottel for his interest and suggestions during the preparation of this manuscript.

NOTATION

C_γ = universal constant
d_N = nozzle diameter in ducted jet experiment, L
D = molecular diffusivity, $L^2 - t^{-1}$
g_c = conversion constant in Newton's law, $MLF^{-1} t^{-2}$
$G_1(k_1)$ = unnormalized spectrum. $\int_0^\infty G_1(k)\, dk = \gamma'^2$, L
k_1 = wave number for the one-dimensional spectrum $G_1(k_1)$, L^{-1}
k_o = characteristic wave number of big eddies, L^{-1}
u' = root-mean-square velocity fluctuation, Lt^{-1}
M = mesh dimension of a wind tunnel grid, L
N_{Re} = Reynolds number
N_{Sc} = Schmidt number
N_{Th} = throttling number [Defined by Equation (16)]
$\dot{v}_T$ = total volumetric feed rate to ducted jet mixer, $L^3\, t^{-1}$
r_N = nozzle radius in ducted jet mixer, L
r_D = cylindrical duct radius in Becker's experiments, L
u_N = nozzle fluid discharge velocity in ducted jet mixer, Lt^{-1}
u_I = inspired fluid average velocity in ducted jet mixer, Lt^{-1}
V = holdup volume

Greek Letters

β = proportionality constant relating k_o and d^{-1}
γ = local fluctuation from the mean concentration
$\bar{\Gamma}$ = mean concentration, volume fraction
γ' = root-mean-square of γ
ϵ = mechanical energy cascade rate, $-\frac{3}{2}\frac{1}{g_c}\frac{du'^2}{dt}$, $FLt^{-1}M^{-1}$
ϵ_γ = concentration energy cascade rate, $-\frac{d\gamma'^2}{dt}$, t^{-1}
η = fraction of energy that cascades the turbulence spectrum
η_v = Kolmogoroff microscale of the velocity field, L
η_γ = Kolmogoroff microscale of the concentration field, L
Λ_γ = integral scale of random concentration field, L
θ = root-mean-square temperature fluctuation, θ
τ = mean residence time, t

Measurements

F = force
L = length
M = mass
t = time
θ = temperature

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Manuscript received May 17, 1962; revision received April 24, 1963; paper accepted June 4, 1963.